

Analytical Expression of the Expected Values of Capital at Voting in the Stochastic Environment

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Abstract—In the simplest variant of the model of collective decision making in the stochastic environment, the participants were segregated into egoists and a group of collectivists. “Proposal of the environment” is the stochastically generated vector of algebraic increments of capitals. The social dynamics was defined by the sequence of proposals accepted by threshold-majority voting. Analytical expressions of the expected values of the capitals of participants, collectivists and egoists were obtained. Distinctions of some principles of group voting were discussed.

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1. INTRODUCTION

A model where the participants vote for the projects of redistribution of their own property was analyzed by A.V. Malishevskii in the late 1960’s [1, pp. 93–95]. In this model, voting is greatly manipulatable by the organizers, as it is the case with the participants whose “ideals” are the points in the multidimensional space of programs [2] (see also [3]). The monograph [4] is devoted to the spatial voting models. The problems of relations between selfishness, altruism, and rationality were considered in [5–7], and voting as a method of making decisions about redistribution of social benefits by means of taxation and social programs was discussed in [8–11].

For a stochastic environment oriented to the analysis of “effectiveness” of the voters’ collectivist and selfish attitudes in the conditions where new programs are randomly generated by the “environment,” rather than developed by the organizers or participants of voting, a model of voting was suggested by the present author and analyzed in [12]. Consideration was given to the case of neutral environment, the rest of the cases being analyzed mostly in qualitative terms. Emphasis was made on the time dependencies of the participant “capitals” under various values of the model parameters. In what follows, the present author obtained explicit expressions of these dependencies, the exact formulas of expectations and their normal approximations. These formulas enable one to determine the nature of social dynamics under any values of the model parameters. Expressions for the expectations of the capital increments were obtained in Sec. 3, the necessary lemmas being proved in Sec. 2.

We give a thumbnail of the simplest variant of the model. The “society” is assumed to consist of n “participants” of which ℓ are “egoists” and $g = n - \ell$ are the “group” members. At any time instant, the participant is characterized by its “capital” expressed by a real number and interpreted in the most general sense (like *utility*). Each participant has some starting capital. The “environment proposal” is the vector (d_1, \dots, d_n) of the algebraic increments of the capitals of all participants. These increments are independent-in-aggregate random variables with identical distribution $N(\mu, \sigma^2)$, where μ and σ are the model parameters. At each step, one random “environment proposal” is put to the vote. Each “egoist” votes “for” if and only if the proposal brings it

a positive capital increment. Each member of the group votes “for” if and only if the group gains from the realization of this proposal. The “gain” may be understood differently. The basic model considers two fundamental principles of decision voting .

Principle A. *The group votes “for” the proposal (d_1, \dots, d_n) if and only if as the result of its approval the number of group members getting a positive capital increment exceeds that of the group members getting a negative increment.*

Principle B. *The group votes “for” the proposal (d_1, \dots, d_n) if and only if the sum of increments of group members is positive: $\sum d_i > 0$, where the sum is taken over the subscripts of the group participants.*

The results of voting are summarized using the “ α -majority” procedure: the proposal is accepted if and only if more than αn , $0 \leq \alpha < 1$, participants vote for it. If the proposal is accepted, then the capitals of all participants get the corresponding increments d_1, \dots, d_n , or remain the same, otherwise.

Capital dynamics of the participants is analyzed in terms of their social roles (“egoist” or “group member”) and model parameters. It is implied, in particular, that a scenario is plausible where the egoists join the group and the “group egoism” resembles more and more the decision making in the interests of the entire society. By another hypothetical scenario, an “ineffective” group dissolves, and its members either become “egoists” or make other groups. It is planned also to consider the case of a socially oriented group supporting its poorest members and preventing their ruin, a variant of the model where the capital increments depend on their current values, the impact on the social dynamics of the mechanisms of taxation and collection of the “party dues,” and so on. At the same time, it is not planned to consider purely economical mechanisms of capital reproduction and loss because our aim lies in analyzing the social, rather than the economic phenomena.

Consideration is given not only to the traditional decision threshold $\alpha = 0.5$ corresponding to the “simple majority,” but to all thresholds ranging from 0 to 1. That is due to the fact that the most important—for example, “constitutional”—decisions are accepted by the a “qualified parliamentary majority” with a threshold greater than 0.5. On the other hand, there are “initiative” decisions such as forming new deputy groups in parliament, putting question on agenda, sending requests to other state authorities, initiating referendums, and so on which can be approved by a certain number of votes smaller than one half. This model is discussed in more detail in [12].

The relation between the numbers of egoists and group members is defined by the parameter $\beta = \ell/2n$, half of the portion of egoists among all participants. We denote by $f_{\mu,\sigma}(\cdot)$ and $F_{\mu,\sigma}(\cdot)$, respectively, the one-dimensional density and the cumulative distribution function corresponding to the distribution $N(\mu, \sigma^2)$; $f(\cdot)$ and $F(\cdot)$ stand for density and the distribution function of the normal distribution with center at 0 and variance 1; $M(\xi)$ and $\sigma(\xi)$ are, respectively, the expectation and the deviation of any of the random variable ξ at hand.

Each i th egoist participant votes for a proposal if and only if $d_i > 0$. The probability of this event is as follows:

$$p = P\{d_i > 0\} = 1 - F_{\mu,\sigma}(0) = F\left(\frac{\mu}{\sigma}\right); \quad (1)$$

the probability of voting “against” the proposal is as follows:

$$q = 1 - p = P\{d_i < 0\} = F_{\mu,\sigma}(0) = 1 - F\left(\frac{\mu}{\sigma}\right) = F\left(-\frac{\mu}{\sigma}\right). \quad (2)$$

In what follows, we also need the notation

$$f = f\left(\frac{\mu}{\sigma}\right). \quad (3)$$

According to the model, the probability that a participant abstains from voting is zero because the normal distribution is continuous. Therefore, the voting of an egoist participant is the Bernoulli test with the parameter p . Then, since the values d_i are independent, the number of egoists voting “for” is distributed *binomially* with the parameters ℓ and p . The mean value and the variance of this distribution are, respectively, $p\ell$ and $pq\ell$.

2. LEMMAS OF THE “NORMAL VOTING SAMPLE”

In this section we prove lemmas that underlie the following calculations. By the “*normal voting sample*” of size¹ ℓ with the parameters (μ, σ^2) and voting threshold ℓ_0 is meant the totality of random variables $(\zeta_1 I(\bar{\zeta}, \ell_0), \dots, \zeta_\ell I(\bar{\zeta}, \ell_0))$, where $\bar{\zeta} = (\zeta_1, \dots, \zeta_\ell)$ is a sample from the distribution $N(\mu, \sigma^2)$,

$$I(\bar{\zeta}, \ell_0) = \begin{cases} 1 & \text{if } n^+(\bar{\zeta}) > \ell_0; \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

and²

$$n^+(\bar{\zeta}) = \#\{k : \zeta_k > 0, k = 1, \dots, \ell\}. \quad (5)$$

The following lemma holds.

Lemma 1. *Let $(\eta_1, \dots, \eta_\ell)$ be a normal voting sample with the parameters (μ, σ^2) and the voting threshold ℓ_0 . Then,³ for any $k = 1, \dots, \ell$*

$$M(\eta_k) = \sum_{x=[\ell_0]+1}^{\ell} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) \binom{\ell}{x} p^x q^{\ell-x}, \quad (6)$$

where p , q , and f are defined in (1)–(3).

The proofs are given in the Appendix.

To calculate (6), one may use the well-known relation between the binomial distribution and the beta-distribution:

$$\sum_{x=t}^{\ell} \binom{\ell}{x} p^x q^{\ell-x} = B(p \mid t, \ell - t + 1), \quad (7)$$

the right-hand side contains the cumulative function of the distribution of beta-distribution with t and $\ell - t + 1$ degrees of freedom. However, the normal approximation of the binomial probability can be used even for a comparatively small ℓ :

$$\binom{\ell}{x} p^x q^{\ell-x} \approx f \left(\frac{x - \mu'}{\sigma'} \right) \approx F \left(\frac{x + 0.5 - \mu'}{\sigma'} \right) - F \left(\frac{x - 0.5 - \mu'}{\sigma'} \right), \quad (8)$$

where μ' and σ' are the same parameters as for the binomial distribution: $\mu' = p\ell$, $\sigma' = \sqrt{pq\ell}$. Summation in (8) provides an approximation to the binomial distribution function:

$$\begin{aligned} \sum_{x=0}^t \binom{\ell}{x} p^x q^{\ell-x} &\approx F \left(\frac{t + 0.5 - p\ell}{\sqrt{pq\ell}} \right) \quad \text{and} \\ \sum_{x=t+1}^{\ell} \binom{\ell}{x} p^x q^{\ell-x} &\approx F \left(-\frac{t + 0.5 - p\ell}{\sqrt{pq\ell}} \right). \end{aligned} \quad (9)$$

¹ In this section, ℓ is an arbitrary integer.

² $\#X$ denotes the number of elements in the finite set X .

³ Here and below the sum is zero if the lower limit is greater than the upper one. The integer part is bracketed.

The normal approximation is advisable for $pq\ell \geq 9$. For a fixed $pq\ell$, its accuracy is maximal for $p = 0.5$ and decreases with p approaching 0 or 1. Therefore, the normal approximation is often recommendable for $0.1 < p < 0.9$ already if $pq\ell > 5$. For the values of p that are very close to 0 or 1, usually the condition $pq\ell > 25$ is imposed.

Lemma 2. *Let $(\eta_1, \dots, \eta_\ell)$ be a normal voting sample with the parameters (μ, σ^2) and the voting threshold ℓ_0 . Then substitution of the standard normal approximation of the random variable $n^+(\bar{\zeta})$ for its binomial distribution provides the following approximation of $M(\eta_k)$ for any $k = 1, \dots, \ell$:*

$$M(\eta_k) \approx \mu F(-\ell'_0) + \frac{\sigma f}{\sqrt{pq\ell}} f(\ell'_0), \quad (10)$$

where $\ell'_0 = ([\ell_0] + 0.5 - p\ell)/\sqrt{pq\ell}$.

This approximation is equivalent to the replacement of a finite number of the participants of voting by a continuous “voting field.” We present for completeness sake the following lemma where $J(\bar{\zeta}, \ell_0)$ differs from $I(\bar{\zeta}, \ell_0)$ (4) in the sign of inequality in the definition. Its proof is similar to that of Lemmas 1 and 2.

Lemma 3. *Let $(\eta_1, \dots, \eta_\ell) = (\zeta_1 J(\bar{\zeta}, \ell_0), \dots, \zeta_\ell J(\bar{\zeta}, \ell_0))$, where $\bar{\zeta} = (\zeta_1, \dots, \zeta_\ell)$, be a sample from the distribution $N(\mu, \sigma^2)$ and*

$$J(\bar{\zeta}, \ell_0) = \begin{cases} 1 & \text{if } n^+(\bar{\zeta}) \leq \ell_0; \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Then, for any $k = 1, \dots, \ell$

$$M(\eta_k) = \sum_{x=0}^{[\ell_0]} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) \binom{\ell}{x} p^x q^{\ell-x}, \quad (12)$$

and the normal approximation provides the approximation

$$M(\eta_k) \approx \mu F(\ell'_0) - \frac{\sigma f}{\sqrt{pq\ell}} f(\ell'_0), \quad (13)$$

where $\ell'_0 = ([\ell_0] + 0.5 - p\ell)/\sqrt{pq\ell}$.

To sum up the auxiliary results, we denote by $\mu^+(\mu, \sigma, \ell, \ell_0)$ the expectation of the normal voting sample of size ℓ with the parameters (μ, σ^2) and the voting threshold ℓ_0 , and by $\mu^-(\mu, \sigma, \ell, \ell_0)$, the expectation of the set of random variables as defined in Lemma 3. Then, the exact and approximate values of $\mu^+(\mu, \sigma, \ell, \ell_0)$ obey the formulas (6) and (10), and the exact and approximate values of $\mu^-(\mu, \sigma, \ell, \ell_0)$ obey the formulas (12) and (13).

3. GENERAL EXPRESSIONS FOR THE MEAN INCREMENTS OF CAPITALS

3.1. Increment of the Egoist's Capital

Let us calculate the expectation $M(\tilde{d}_{\mathcal{E}})$ of the egoist's capital increment in one step.⁴ We recall that a step lies in considering one proposal of the environment by the participants, independently of

⁴ The notation under tilde refers to the actual, in contrast to the proposed, capital increments.

whether it is accepted or not. Let the event G be supported by the group, and \overline{G} be the opposite. Then,

$$M(\tilde{d}_{\mathcal{E}}) = M(\tilde{d}_{\mathcal{E}} | G)P\{G\} + M(\tilde{d}_{\mathcal{E}} | \overline{G})P\{\overline{G}\}. \quad (14)$$

We denote $P\{G\}$ and $P\{\overline{G}\} = 1 - P\{G\}$, respectively, by P_G and Q_G .

Support of a proposal by the group brings it $(1 - 2\beta)n$ votes. Consequently, for approval of a proposal on condition that it is supported by the group, it is necessary and sufficient that more than $(\alpha - (1 - 2\beta))n$ egoists vote for it. If the group does not support the proposal, then it is necessary and sufficient that more than αn egoists vote for it. Therefore,

$$M(\tilde{d}_{\mathcal{E}} | G) = \mu^+(\mu, \sigma, \ell, \gamma n), \quad M(\tilde{d}_{\mathcal{E}} | \overline{G}) = \mu^+(\mu, \sigma, \ell, \alpha n), \quad (15)$$

where $\gamma = \alpha - (1 - 2\beta)$ and $[t]$ is the integer part of t . Hence,

$$M(\tilde{d}_{\mathcal{E}}) = \mu^+(\mu, \sigma, \ell, \gamma n)P_G + \mu^+(\mu, \sigma, \ell, \alpha n)Q_G. \quad (16)$$

By substituting expression (6) for $\mu^+(\cdot, \cdot, \cdot, \cdot)$ in (16) and using the notation

$$b(x | \ell) = \binom{\ell}{x} p^x q^{\ell-x}, \quad (17)$$

we obtain

$$\begin{aligned} M(\tilde{d}_{\mathcal{E}}) &= P_G \sum_{x=[\gamma n]+1}^{\ell} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) b(x | \ell) \\ &+ Q_G \sum_{x=[\alpha n]+1}^{\ell} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) b(x | \ell) \\ &= P_G \sum_{x=[\gamma n]+1}^{[\alpha n]} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) b(x | \ell) \\ &+ \sum_{x=[\alpha n]+1}^{\ell} \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{p\ell} - 1 \right) \right) b(x | \ell). \end{aligned} \quad (18)$$

If the group follows Principle A, then

$$P_G^A = \sum_{x=[g/2]+1}^g b(x | g) \quad \text{and} \quad Q_G^A = 1 - P_G^A = \sum_{x=0}^{[g/2]} b(x | g) \quad (19)$$

should be substituted for P_G and Q_G , respectively, and in the case of Principle B,

$$P_G^B = F\left(\frac{\mu\sqrt{g}}{\sigma}\right) \quad \text{and} \quad Q_G^B = 1 - P_G^B = F\left(-\frac{\mu\sqrt{g}}{\sigma}\right), \quad (20)$$

because for an arbitrary proposal the mean capital increment of the group has the distribution $N\left(\mu, \frac{\sigma}{\sqrt{g}}\right)$.

Now, by substituting in (16) the normal approximation (10) for $\mu^+(\cdot, \cdot, \cdot, \cdot)$ and using the notation

$$F_{\theta} = F\left(-\frac{[\theta n] + 0.5 - p\ell}{\sqrt{pq\ell}}\right), \quad f_{\theta} = f\left(\frac{[\theta n] + 0.5 - p\ell}{\sqrt{pq\ell}}\right), \quad (21)$$

we get

$$\begin{aligned} M(\tilde{d}_{\mathcal{E}}) &\approx P_G \left(\mu F_{\gamma} + \frac{\sigma f}{\sqrt{pq\ell}} f_{\gamma} \right) + Q_G \left(\mu F_{\alpha} + \frac{\sigma f}{\sqrt{pq\ell}} f_{\alpha} \right) \\ &= [P_G \ Q_G] \left(\mu \begin{bmatrix} F_{\gamma} \\ F_{\alpha} \end{bmatrix} + \frac{\sigma f}{\sqrt{pq\ell}} \begin{bmatrix} f_{\gamma} \\ f_{\alpha} \end{bmatrix} \right) \end{aligned} \quad (22)$$

where matrix notation is used. In the case of Principle A, the approximate values

$$\begin{aligned} F \left(-\frac{[\frac{g}{2}] + 0.5 - pg}{\sqrt{pqg}} \right) &\approx P_G^A, \\ 1 - F \left(-\frac{[\frac{g}{2}] + 0.5 - pg}{\sqrt{pqg}} \right) &= F \left(\frac{[\frac{g}{2}] + 0.5 - pg}{\sqrt{pqg}} \right) \approx Q_G^A \end{aligned} \quad (23)$$

or exact values (19) must be substituted in this formula; in the case of Principle B, the exact values of (20).

3.2. Increment of the Capital of Group Member

Now we derive formulas similar to (18) and (22) for the expectation $M(\tilde{d}_{\mathcal{G}})$ of the capital increment of a group member in one step. Let E_{θ} be an event of egoists giving more than θn votes to the proposal; \overline{E}_{θ} , otherwise. If E_{α} is satisfied, then the proposal is accepted independently of the group voting; if $\overline{E}_{\alpha} \wedge E_{\gamma}$ takes place, then acceptance/rejection of the proposal is defined by its support by the group. If \overline{E}_{γ} is realized, then the proposal cannot be accepted. Therefore,

$$M(\tilde{d}_{\mathcal{G}}) = M(\tilde{d}_{\mathcal{G}} \mid E_{\alpha}) P\{E_{\alpha}\} + M(\tilde{d}_{\mathcal{G}} \mid \overline{E}_{\alpha} \wedge E_{\gamma}) P\{\overline{E}_{\alpha} \wedge E_{\gamma}\} + 0 \cdot P\{\overline{E}_{\gamma}\}. \quad (24)$$

If E_{α} is satisfied, then the proposal is always accepted. Therefore, $M(\tilde{d}_{\mathcal{G}} \mid E_{\alpha}) = \mu$ by virtue of independence of the capital increments. If $\overline{E}_{\alpha} \wedge E_{\gamma}$ is satisfied, then acceptance of the proposal is defined by the group voting, and generally $M(\tilde{d}_{\mathcal{G}} \mid \overline{E}_{\alpha} \wedge E_{\gamma}) \neq \mu$.

We make use of the notation

$$P_{\theta} = P\{E_{\theta}\} = \sum_{x=[\theta n]+1}^{\ell} b(x \mid \ell) \approx F_{\theta} = F \left(-\frac{[\theta n] + 0.5 - p\ell}{\sqrt{pq\ell}} \right), \quad (25)$$

to rearrange (24) in

$$M(\tilde{d}_{\mathcal{G}}) = \mu P_{\alpha} + M(\tilde{d}_{\mathcal{G}} \mid \overline{E}_{\alpha} \wedge E_{\gamma})(P_{\gamma} - P_{\alpha}). \quad (26)$$

If the group follows Principle A, then $M(\tilde{d}_{\mathcal{G}} \mid \overline{E}_{\alpha} \wedge E_{\gamma}) = \mu^{+}(\mu, \sigma, g, g/2)$, the substitution of (6) in

$$M(\tilde{d}_{\mathcal{G}}^A) = \mu P_{\alpha} + \mu^{+}(\mu, \sigma, g, g/2)(P_{\gamma} - P_{\alpha}), \quad (27)$$

provides

$$\begin{aligned} M(\tilde{d}_{\mathcal{G}}^A) &= \mu P_{\alpha} + \sum_{x=[g/2]+1}^g \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{pg} - 1 \right) \right) b(x \mid g)(P_{\gamma} - P_{\alpha}) \\ &= \mu \sum_{x=[\alpha n]+1}^{\ell} b(x \mid \ell) + \sum_{x=[g/2]+1}^g \left(\mu + \frac{\sigma f}{q} \left(\frac{x}{pg} - 1 \right) \right) b(x \mid g) \sum_{x=[\gamma n]+1}^{[\alpha n]} b(x \mid \ell). \end{aligned} \quad (28)$$

In order to obtain the normal approximation of this value, we replace $\mu^+(\mu, \sigma, g, g/2)$ in (27) by the approximation (10), and P_α and P_γ , by the approximations F_α and F_γ (see (25)). By using the notation

$$f_G^A = f \left(\frac{\lfloor \frac{g}{2} \rfloor + 0.5 - pg}{\sqrt{pqg}} \right), \quad (29)$$

we obtain

$$\begin{aligned} M(\tilde{d}_G^A) &\approx \mu F_\alpha + \left(\mu P_G^A + \frac{\sigma f f_G^A}{\sqrt{pqg}} \right) (F_\gamma - F_\alpha) \\ &= \mu (P_G^A F_\gamma + (1 - P_G^A) F_\alpha) + \frac{\sigma f f_G^A}{\sqrt{pqg}} (F_\gamma - F_\alpha) \\ &= [F_\gamma \ F_\alpha] \left(\mu \begin{bmatrix} P_G^A \\ Q_G^A \end{bmatrix} + \frac{\sigma f f_G^A}{\sqrt{pqg}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \end{aligned} \quad (30)$$

If the group follows Principle B, then $M(\tilde{d}_G | \bar{E}_\alpha \wedge E_\gamma) = \mu^+ \left(\mu, \frac{\sigma}{\sqrt{g}}, 1, 0 \right)$, and (26) is reshaped in

$$M(\tilde{d}_G^B) = \mu P_\alpha + \mu^+ \left(\mu, \frac{\sigma}{\sqrt{g}}, 1, 0 \right) (P_\gamma - P_\alpha). \quad (31)$$

To determine $\mu^+ \left(\mu, \frac{\sigma}{\sqrt{g}}, 1, 0 \right)$, one may substitute the corresponding arguments in (6) or, which is even simpler, multiply (A.4) by p :

$$\mu^+ \left(\mu, \frac{\sigma}{\sqrt{g}}, 1, 0 \right) = \mu F \left(\frac{\mu \sqrt{g}}{\sigma} \right) + \frac{\sigma}{\sqrt{g}} f \left(\frac{\mu \sqrt{g}}{\sigma} \right). \quad (32)$$

By using the notation (20) and

$$f_G^B = f \left(\frac{\mu \sqrt{g}}{\sigma} \right), \quad (33)$$

we obtain

$$\begin{aligned} M(\tilde{d}_G^B) &\approx \mu P_\alpha + \left(\mu P_G^B + \frac{\sigma f_G^B}{\sqrt{g}} \right) (P_\gamma - P_\alpha) \\ &= [P_\gamma \ P_\alpha] \left(\mu \begin{bmatrix} P_G^B \\ Q_G^B \end{bmatrix} + \frac{\sigma f_G^B}{\sqrt{g}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right), \end{aligned} \quad (34)$$

where either the exact values or normal approximations (see (25)) can be substituted for P_γ and P_α . The expected values of the capital increments of the egoists and members of the group in a series of s steps obey, respectively, $sM(\tilde{d}_\mathcal{E})$ and $sM(\tilde{d}_\mathcal{G})$.

3.3. Voting Principles A' and A''

In addition to Principles A and B, the group can make use of Principle A' [12] enabling it sometimes to minimize and sometimes to eliminate the advantage of the egoists in those zones of variation of the threshold α where they have this advantage. This is reached at the expense of

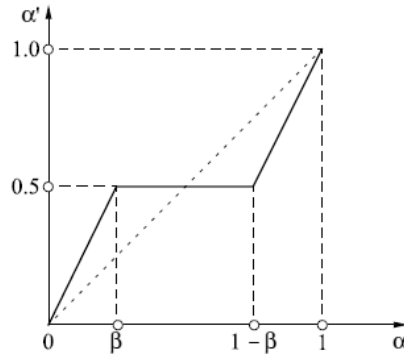


Fig. 1. Threshold α' vs. α .

reduced group capital increment in these zones: relative gain of the group leads to absolute losses. We formulate below this voting principle. The threshold α' vs. α and β is depicted in Fig. 1.

Principle A'. *The group votes “for” the proposal of the environment if and only if as the result of accepting it the part of its members getting a positive capital increment exceeds the threshold*

$$\alpha' = \begin{cases} \frac{1}{2} - \frac{\delta}{2\beta}, & \text{where } \delta = \beta - \alpha, & \text{for } \alpha < \beta; \\ \frac{1}{2} + \frac{\delta}{2\beta}, & \text{where } \delta = \alpha - (1 - \beta), & \text{for } \alpha > 1 - \beta; \\ \frac{1}{2} & & \text{for } \alpha \in [\beta, 1 - \beta] \end{cases} \quad (35)$$

$$= \begin{cases} \frac{\alpha}{2\beta} & \text{for } \alpha < \beta; \\ 1 - \frac{1-\alpha}{2\beta} & \text{for } \alpha > 1 - \beta; \\ \frac{1}{2} & \text{for } \alpha \in [\beta, 1 - \beta]. \end{cases}$$

Since Principle A' differs from Principle A only in the internal threshold of group voting, for calculation of the expected capital increments it suffices to replace $[g/2]$ by $[\alpha'g]$ in (19), (23), (28), and (29). Thus, in (18), (22), and (30) one has to substitute

$$P_G^{A'} = \sum_{x=[\alpha'g]+1}^g b(x | g) \quad (36)$$

$$\text{or the approximation } F\left(-\frac{[\alpha'g] + 0.5 - pg}{\sqrt{pqg}}\right) \approx P_G^{A'},$$

for P_G and P_G^A ,

$$Q_G^{A'} = 1 - P_G^{A'} = \sum_{x=0}^{[\alpha'g]} b(x | g) \quad (37)$$

$$\text{or the approximation } F\left(\frac{[\alpha'g] + 0.5 - pg}{\sqrt{pqg}}\right) \approx Q_G^{A'},$$

for Q_G and Q_G^A , and

$$f_G^{A'} = f\left(\frac{[\alpha'g] + 0.5 - pg}{\sqrt{pqg}}\right) \quad (38)$$

for f_G^A .

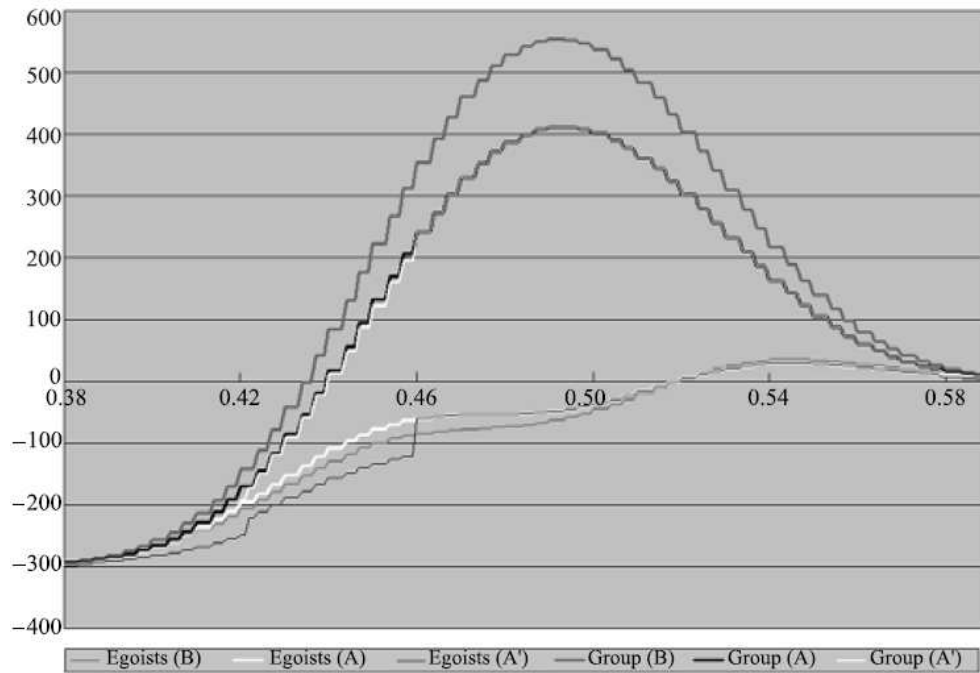


Fig. 2. Capital increments of the egoists and group members following Principles A, B, and A' vs. the decision threshold α . Values of the parameters: $\beta = 0.46$; $n = 300$; $\mu = -0.3$; $\sigma = 10$; $s = 1000$.

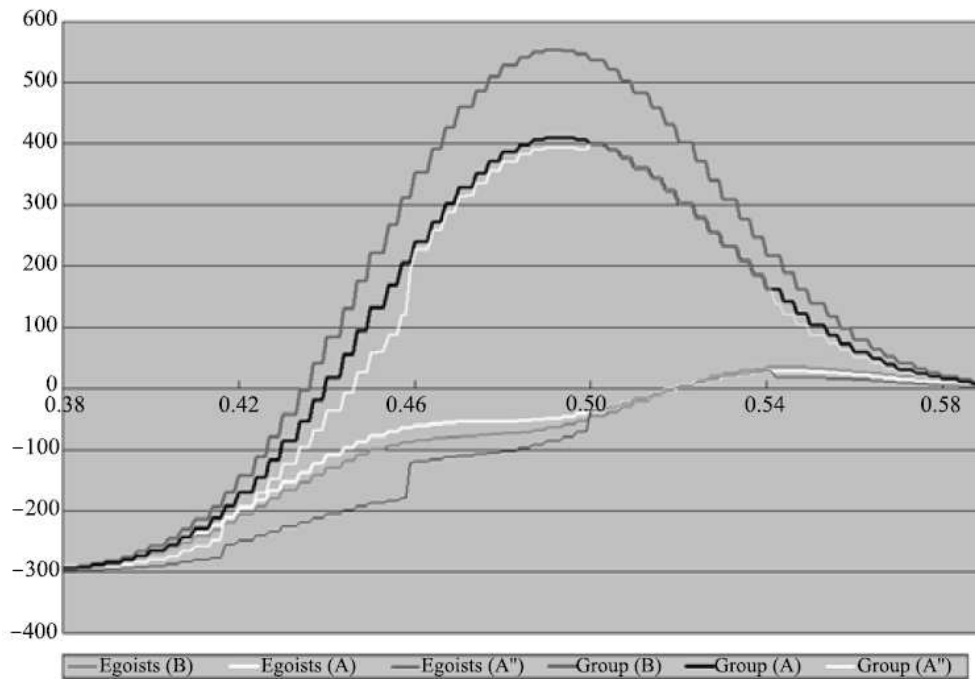


Fig. 3. Capital increments of egoists and group members following Principles A, B, and A'' vs. the decision threshold α . Values of the parameters: $\beta = 0.46$; $n = 300$; $\mu = -0.3$; $\sigma = 10$; $s = 1000$.

We present some examples of applying the above formulas. For $\mu = -0.3$, $\sigma = 10$, $2\beta = 0.92$, $n = 300$ participants, and $s = 1000$ steps, Fig. 2 shows the expected capital increments of the

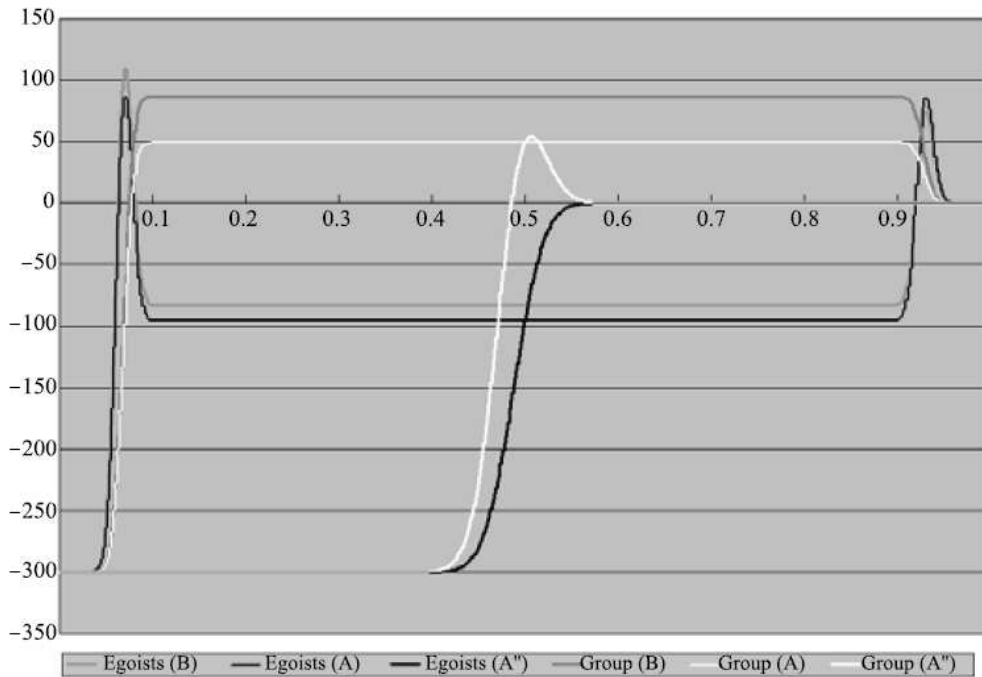


Fig. 4. Capital increments of the egoists and group members following Principles A, B, and A'' vs. the decision threshold α . Values of the parameters: $\beta = 0.07$; $n = 450$; $\mu = -0.3$; $\sigma = 10$; $s = 1000$.

egoists and group members vs. the decision threshold, provided that the group makes use of different voting principles.

The functions in Fig. 2 are stepwise because the results of voting are defined by the ratio of the number of participant voting "for" and the integer part of the threshold αn . The capital increment of a group member in the case of Principle A' is shown by the white line which for $\alpha < 0.46$ lies slightly below the bold black line corresponding to the capital increment of a group member in the case of Principle A. At the same time, at the passage to A' the egoists lose more: for $\alpha < 0.46$, the thin black line that represents the variation of their mean capital passes much lower the bold white line corresponding to Principle A. The capital of a group member grows to the limit for $\alpha \approx 0.494$ by 410 units, whereas in the case of accepting all proposals the reduction in capital is 300 units. The expected capital increment of the egoist is positive for $\alpha > 0.52$ and maximal for $\alpha \approx 0.545$. The group can use a more radical approach to the choice of the internal threshold assuming that it is equal to the external one.

Principle A''. *The group votes "for" a proposal of the environment if and only if as the result of accepting it the part of its members getting a positive capital increment exceeds α .*

In this case, the capital increments are calculated as in the case of Principle A' with $[\alpha g]$ in place of $[\alpha' g]$. An example of using Principle A'' is depicted in Fig. 3, all parameters being the same as in Fig. 2.

The passage from Principle A' to Principle A'' in the main worsens the group activities and, additionally, changes completely the dynamics of the egoists' capital. In the example of Fig. 4, practically all proposals of the unfavorable environment are accepted for $\alpha < 0.4$, and practically no proposal is accepted for $\alpha > 0.57$.

For $\alpha \approx 0.508$, the expected capital increment of a group member is maximal: $M(\tilde{d}_G^{A''}) \approx 54$; the capital increment of the egoist does not become positive under any α .

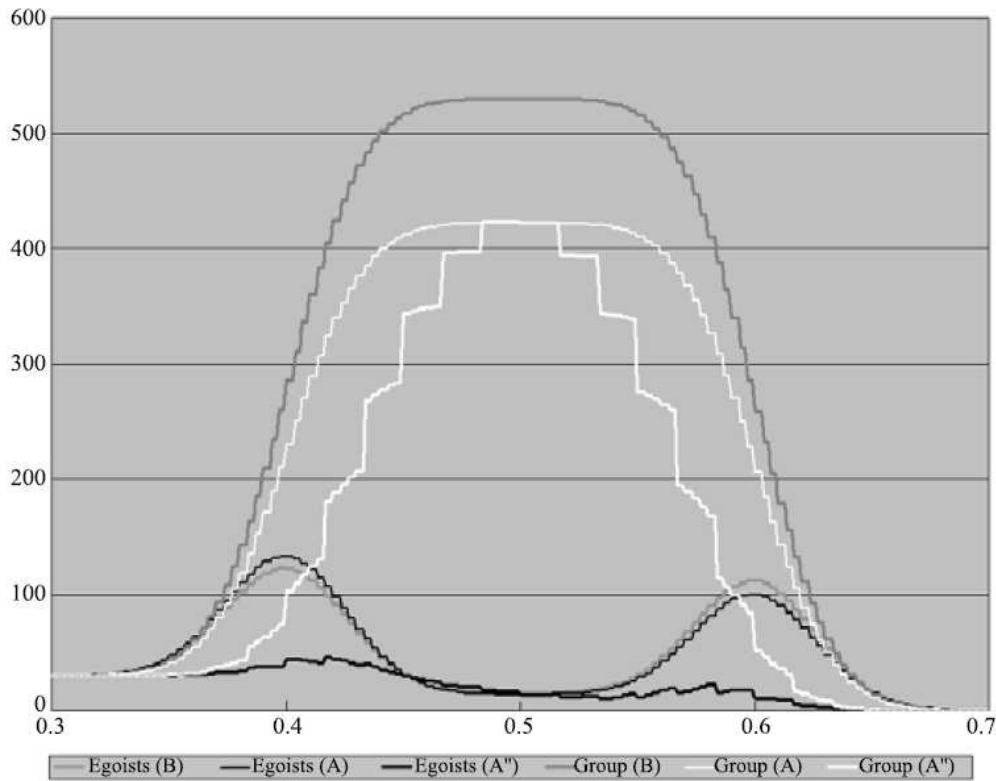


Fig. 5. Capital increments of the egoists and group members following Principles A, B, and A'' vs. the decision threshold α . Values of the parameters: $\beta = 0.4$; $n = 300$; $\mu = 0.03$; $\sigma = 10$; $s = 1000$.

The main distinction of Principle A'' lies in that it leads to actual disappearance of the maxima of capital increments of the egoists, provided that their fraction is excessive. In the case of Principles A and B, these peaks are especially high in zones 4 and 5 namely for a small number of egoists (see Fig. 4), and the passage to Principle A'' eliminates them. For 300 participants, the height of these peaks is negligible if the egoists make up less than two thirds of the participants; low peaks for 80% of egoists (low “strongly plowed hills”) can be seen in Fig. 5.

If an optimality criterion is defined, then one may raise the question of the optimal choice of the intragroup voting threshold. Similarly, it is possible to vary the threshold of proposal support for principles like B. In the case of favorable environment, in particular, the requirements on the mean one-step capital increment of a group member can be set at a higher level than in an unfavorable environment. All examples show that to a certain extent the group can “dictate its will.” It deserves noting that for 300 participants, neutral environment, and simple-majority procedure, the ratio of the expected capital increment of the member of a three-participant group to the capital increment of an egoist is approximately 1.4 in the case of Principle A and 1.75 in the case of Principle B.

4. CONCLUSIONS

Exact and approximate (based on the normal approximation of the binomial distribution) formulas for the expectations of the capital increment of the “egoists” and the “group members” were obtained for the considered model of voting in an environment with certain random parameters. These formulas enable one to determine the form of the model trajectory for any values of the parameters. The distinctions of different voting principles were considered using several examples.

Proof of Lemma 1. Let $(\eta_1, \dots, \eta_\ell) = (\zeta_1 I(\bar{\zeta}, \ell_0), \dots, \zeta_\ell I(\bar{\zeta}, \ell_0))$. We denote $n^+(\bar{\zeta})$ by n^+ for brevity. Using the formula of total probability for expectation, we get

$$M(\eta_k) = 0 \cdot P\{n^+ \leq \ell_0\} + \sum_{x=[\ell_0]+1}^{\ell} M(\eta_k | n^+ = x) P\{n^+ = x\} \quad (\text{A.1})$$

for any $k = 1, \dots, \ell$. Then

$$\begin{aligned} M(\eta_k | n^+ = x) &= M(\eta_k | n^+ = x, \eta_k > 0) P\{\eta_k > 0 | n^+ = x\} \\ &+ M(\eta_k | n^+ = x, \eta_k \leq 0) P\{\eta_k \leq 0 | n^+ = x\}. \end{aligned} \quad (\text{A.2})$$

It follows from independence of the sample elements that

$$M(\eta_k | n^+ = x, \eta_k > 0) = M(\eta_k | \eta_k > 0) = M(\zeta_k | \zeta_k > 0). \quad (\text{A.3})$$

This expression is the mean normal random variable, if positive, which is determined by integration and is as follows:

$$M(\zeta_k | \zeta_k > 0) = p^{-1} \frac{1}{\sqrt{2\pi}\sigma} \int_0^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu + \frac{\sigma f}{p}, \quad (\text{A.4})$$

where $f = f(\mu/\sigma)$. Similarly,

$$M(\eta_k | n^+ = x, \eta_k \leq 0) = M(\zeta_k | \zeta_k \leq 0) = \mu - \frac{\sigma f}{q}. \quad (\text{A.5})$$

We note that $P\{\eta_k > 0 | n^+ = x\} = x/\ell$ and $P\{\eta_k \leq 0 | n^+ = x\} = 1 - x/\ell$, substitute the determined values in (A.2)

$$M(\eta_k | n^+ = x) = \left(\mu + \frac{\sigma f}{p}\right) \frac{x}{\ell} + \left(\mu - \frac{\sigma f}{q}\right) \left(1 - \frac{x}{\ell}\right) = \mu - \frac{\sigma f}{q} + \frac{x\sigma f}{pq\ell} \quad (\text{A.6})$$

and (A.1)

$$M(\eta_k) = \sum_{x=[\ell_0]+1}^{\ell} \left(\mu - \frac{\sigma f}{q} + \frac{x\sigma f}{pq\ell}\right) \binom{\ell}{x} p^x q^{\ell-x}, \quad (\text{A.7})$$

which completes the proof of Lemma. \square

Proof of Lemma 2. We rearrange (6) in

$$M(\eta_k) = \sum_{x=[\ell_0]+1}^{\ell} \binom{\ell}{x} p^x q^{\ell-x} \left(\mu - \frac{\sigma f}{q} + \frac{\sigma f}{pq\ell} \cdot \frac{\sum_{x=[\ell_0]+1}^{\ell} x \binom{\ell}{x} p^x q^{\ell-x}}{\sum_{x=[\ell_0]+1}^{\ell} \binom{\ell}{x} p^x q^{\ell-x}} \right). \quad (\text{A.8})$$

The first sum in the right-hand side is $P\{n^+ > \ell_0\}$; it is approximated by $F(-\ell'_0)$. The ratio of the two remaining sums is $M(n^+ | n^+ > \ell_0)$. For $\zeta \sim N(\mu, \sigma^2)$,

$$M(\zeta | \zeta > t) = \mu + \sigma \frac{f\left(\frac{\mu-t}{\sigma}\right)}{F\left(\frac{\mu-t}{\sigma}\right)} \quad (\text{A.9})$$

(this formula stems from (A.4)). We use it to approximate $M(n^+ | n^+ > \ell_0)$ and obtain

$$M(n^+ | n^+ > \ell_0) \approx p\ell + \sqrt{pq\ell} \frac{f(\ell'_0)}{F(-\ell'_0)}. \quad (\text{A.10})$$

Substitution of these approximations in (A.8) provides

$$\begin{aligned} M(\eta_k) &\approx F(-\ell'_0) \left(\mu - \frac{\sigma f}{q} + \frac{\sigma f}{pq\ell} \left(p\ell + \sqrt{pq\ell} \frac{f(\ell'_0)}{F(-\ell'_0)} \right) \right) \\ &= \mu F(-\ell'_0) + \frac{\sigma f}{\sqrt{pq\ell}} f(\ell'_0). \end{aligned} \quad (\text{A.11})$$

□

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